Average Collapsibility of Some Association Measures

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Abstract. Collapsibility deals with the conditions under which a conditional (on a covariate *W*) measure of association between two random variables *X* and *Y* equals the marginal measure of association, under the assumption of homogeneity over the covariate. In this paper, we discuss the average collapsibility of certain well-known measures of association, and also with respect to a new measure of association. The concept of average collapsibility is more general than collapsibility, and requires that the conditional average of an association measure equals the corresponding marginal measure. Sufficient conditions for the average collapsibility of the measures under consideration are obtained. Some difficult, but interesting, counter-examples are constructed. Applications to linear, Poisson, logistic and negative binomial regression models are addressed. An extension to the case of multivariate covariate *W* is also discussed.

Key words: Average collapsibility, collapsibility, conditional distributions, linear and non-linear regression models, measures of association, Yule-Simpson paradox.

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1. Introduction

The study of association between two random variables arises in several applications. Several measures, nonparametric in nature, have been proposed in the literature. Often, the random variables of interest, say X and Y, may be associated because of their association with another variable W, called a covariate or a background variable. In this case, we need to investigate the conditional association measure between X and Y given W, and compare it with the marginal association measure between X and Y. It is in general possible that the conditional association measure may be positive, while the marginal association measure may be negative. Such an effect reversal is called the Yule-Simpson paradox attributed to Yule (1903) and Simpson (1951). When Yule-Simpson paradox or the effect reversal does not occur, and a conditional measure of association equals the marginal measure, we say that the measure is collapsible over the covariate W. Collapsibility is an important issue associated with data analysis, analysis of contingency tables, causal inference, regression analysis, epidemiological studies and the design of experiments; see, for example, Cox and Wermuth (2003), Ma et al. (2008), and Xie et al. (2008) for applications and discussions.

There have been several notions of collapsibility, namely, simple, strong and uniform collapsibility. These issues have been addressed in several different contexts such as the analysis of contingency tables, regression models and association measures; see for example Bishop (1971), Cox (2003), Cox and Wermuth (2003), Geng (1992), Ma *et al.* (2006), Vellaisamy and Vijay (2008), Wermuth (1987, 1989), Whittemore (1978), and Xie *et al.* (2008). Cox and Wermuth (2003) studied the concept of distribution dependence and discussed the conditions under which no effect reversal occurs. Xie *et al.* (2008) discussed the simple collapsibility and the uniform collapsibility of the following association measures:

(i)
$$\frac{\partial}{\partial x} E(Y \mid x)$$
 (expectation dependence)
(ii) $\frac{\partial^2}{\partial x \partial y} \log f(x, y)$ (mixed derivative of interaction)
(iii) $\frac{\partial}{\partial x} F(y \mid x)$ (distribution dependence).

They discussed also the stringency of the above measures for positive association, studied the conditions for no effect reversal (after marginalization over *W*) and obtained the necessary and sufficient conditions for uniform collapsibility of mixed derivative of interaction, among other results. Recently, Vellaisamy (2011) introduced a new concept of average collapsibility and discussed it with respect to the distribution dependence and the quantile regression coefficients. It is shown that average collapsibility is a general concept and coincides with collapsibility under the condition of homogeneity. In the same spirit, we discuss in this paper the average collapsibility of expectation dependence, and mixed derivative of interaction measures which have relevance to linear and logistic regression models. Also, a new measure of association, namely,

(iv)
$$\frac{\partial}{\partial x} \log E(Y \mid x)$$
 (log expectation dependence)

is introduced and its average collapsibility conditions are investigated. This measure has a direct application to Poisson and negative binomial regression models. In the last section, some results are extended to the case of multivariate covariate *W*.

2. The Average Collapsibility Results

Let (Y, X, W) be a random vector, where our interest is mainly on the association between Y and X, and W is treated as a covarite. We assume for simplicity that X and W are continuous, unless stated otherwise. Note that Y has a monotone (increasing) regression function of X if E(Y|X=x) is increasing in X or equivalently the expectation dependence function (EDF) $\partial E(Y|X)/\partial X \geq 0$. We first discuss the average collapsibility results for the EDF and introduce the following definition.

Definition 1 The expectation dependence function (EDF) is average collapsible over W if

$$E_{W|x}\left(\frac{\partial}{\partial x}E(Y|x,W)\right) = \frac{\partial}{\partial x}E(Y|x), \quad \text{for all } x. \tag{1}$$

The following result gives sufficient conditions for the average collapsibility of EDF. In the sequel, $X \perp \!\!\!\perp Y$ and $X \perp \!\!\!\perp Y | W$ respectively denote the independence of X and Y, and the conditional independence of X and Y given W. We assume henceforth all the partial derivatives exist and are continuous so that that the differentiation and integration can be interchanged.

Theorem 1 The EDF $\frac{\partial}{\partial x}E(Y|x,w)$ is average collapsible over W if either (i) E(Y|x,w) is independent of w, or (ii) $X \perp \!\!\! \perp W$

holds.

The condition that E(Y|x, w) is independent of w implies the homogeneity of EDF and in this case both uniform collapsibility (Part (a) of Theorem 3.4 of Xie $et\ al.$ (2008)) and average collapsibility hold. However, when the EDF is not homogeneous over w, average collapsibility may still hold if (and only if) $X \perp \!\!\!\perp W$. Observe also that the condition E(Y|x,w) is independent of w is a weaker condition than $Y \perp \!\!\!\perp W|X$, usually required for other notions of collapsibility. For example, when W > 0, and $(Y|x,w) \sim U(x-w,x+w)$, we have E(Y|x,w) = x. for all w. But,

$$F(y|x, w) = \frac{1}{2w}, \quad x - w < y < x + w,$$
(2)

showing that Y and W are not conditionally independent given X.

Some examples for Theorem 1 are the following. Suppose $(W|X=x) \sim N(x,1)$ and $(Y|X=x,W=w) \sim N(x,w)$. As another example, let X>0, $(W|X=x) \sim G(x,1)$ and $(Y|X=x,W=w) \sim G(w,wx)$, where $G(\alpha,p)$ denote the gamma distribution with mean (p/α) . In both the cases, E(Y|x,w)=x is independent of w and so the average collapsibility of $EDF \partial E(Y|x,w)/\partial x$ holds.

We next show that condition (i) or (ii) is only sufficient, but not necessary. Hereafter, $\phi(z)$ and $\Phi(z)$ denote respectively the density and the distribution function of $Z \sim N(0, 1)$.

Example 1 Suppose (Y|X=x, W=w) follows uniform $U(0, (x^2+(w-x)^2))$ so that

$$F(y|x,w) = y(x^2 + (w-x)^2)^{-1}, \quad 0 < y < (x^2 + (w-x)^2)$$
(3)

and $E(Y|x, w) = \frac{1}{2}(x^2 + (w - x)^2)$. Assume also $(W|X = x) \sim N(x, 1)$ so that

$$\frac{\partial}{\partial x}f(w|x) = -\phi'(w-x) = (w-x)\phi(w-x). \tag{4}$$

Then

$$\int E(y|x,w)\frac{\partial}{\partial x}f(w|x)dw = \frac{1}{2}\int_{-\infty}^{\infty} (x^2 + (w-x)^2)(w-x)\phi(w-x)dw$$

$$= \frac{1}{2}\left[x^2\int_{-\infty}^{\infty} (w-x)\phi(w-x)dw + \int_{-\infty}^{\infty} (w-x)^3\phi(w-x)dw\right]$$

$$= \frac{1}{2}\left[x^2\int_{-\infty}^{\infty} t\phi(t)dt + \int_{-\infty}^{\infty} t^3\phi(t)dt\right]$$

$$= 0, \text{ for all } x.$$
(5)

Thus, from (A.2), average collapsibility over W holds, but neither condition (i) nor condition (ii) is satisfied.

We next discuss an implication of Theorem 1 to linear regression models.

Linear regression. Consider the following conditional and marginal linear regression models respectively:

$$E(Y|X = x, W = w) = \begin{cases} \alpha(w) + \beta(w)x, & \text{if } W \text{ is discrete} \\ \alpha + \beta x + \gamma w, & \text{if } W \text{ is continuous} \end{cases}$$

and

$$E(Y|x) = \tilde{\alpha} + \tilde{\beta}x.$$

Then

$$\frac{\partial}{\partial x}E(Y \mid X = x, W = w) = \begin{cases} \beta(w), & \text{if } W \text{ is discrete} \\ \beta, & \text{if } W \text{ is continuous} \end{cases}$$

and

$$\frac{\partial}{\partial x}E(Y\mid x) = \tilde{\beta}.$$

We say that the regression coefficient $\beta(w)$ (or β) is simply collapsible if $\beta(w) = \tilde{\beta}$ for all w (or $\beta = \tilde{\beta}$). Also, it is said to be average collapsible if

$$E_{W|x}(\beta(W)) = \tilde{\beta} \text{ (or } E_{W|x}(\beta) = \tilde{\beta}), \text{ for all } x.$$
 (6)

Thus, the average collapsibility of *EDF* reduces to the average collapsibility of regression coefficients, in the case of linear regression models.

The average collapsibility of regression coefficients $\beta(w)$ under the condition $E_W(\beta(W)) = \tilde{\beta}$ has been discussed by Vellaisamy and Vijay (2007). However, the definition of average collapsibility given in (6) is more natural as it involves the joint distribution of W and X. Note also that $E_{W|x}(\beta(W)) = \tilde{\beta}$ for all x implies $E_W(\beta(W)) = \tilde{\beta}$, but not necessarily conversely.

Next, we look at the average collapsibility of mixed derivative of interaction (MDI). Since

$$\frac{\partial^2}{\partial x \partial y} \log f(x, y) = \frac{\partial^2}{\partial x \partial y} \log f(y|x), \quad \text{for all } x \text{ and } y, \tag{7}$$

it follows from Proposition 3.2.1 of Whittaker (1990) that

$$\frac{\partial^2}{\partial x \partial y} \log f(y|x) = 0 \quad \text{for all } x \text{ and } y \Longleftrightarrow Y \perp\!\!\!\perp X.$$

In view of (7), the *MDI* henceforth stands for $\partial^2 \log f(y|x)/\partial x \partial y$, which motivates the following definition of average collapsibility.

Definition 2 The MDI is said to be average collapsible over W if

$$E_{W|x}\left(\frac{\partial^2}{\partial x \partial y} \log f(y|x, W)\right) = \frac{\partial^2}{\partial x \partial y} \log f(y|x), \quad \text{for all } (y, x).$$

It is assumed that $\log f(y|x)$ has continuous partial derivatives so that

$$\frac{\partial^2}{\partial x \partial y} \log f(y|x) = \frac{\partial^2}{\partial y \partial x} \log f(y|x) \text{ for all } (y, x).$$

The following result provides a set of sufficient conditions for the average collapsibility of MDI.

Theorem 2 The MDI is average collapsible over W if either

- (i) $Y \perp \!\!\!\perp W | X$, or
- $(ii) X \perp \!\!\!\perp W|Y$

holds.

Xie *et al.* (2008)) showed that condition (i) or (ii) in Theorem 2 is necessary and sufficient for uniform collapsibility. The following counter-example shows that they are only sufficient, but not necessary for average collapsibility.

Example 2 Let X > 0 and $(W|x) \sim N(x, 1)$. Assume that

$$f(y|x, w) = xy^{x-1}(x^2 + (w - x)^2), \ 0 < y < (x^2 + (w - x)^2)^{-1/x},$$
(8)

which can easily be seen to be a valid density.

Then

$$\frac{\partial^2}{\partial x \partial y} \log f(y|x, w) = \frac{1}{y} = E_{W|x} \left(\frac{\partial^2}{\partial x \partial y} \log f(y|x, W) \right). \tag{9}$$

Since $(W|x) \sim N(x, 1)$, it follows that the marginal density of (Y|x) is

$$f(y|x) = \int_{-\infty}^{\infty} f(y|x, w) f(w|x) dw$$

= $xy^{x-1} \left[\int_{-\infty}^{\infty} x^2 \phi(w - x) dw + \int_{-\infty}^{\infty} (w - x)^2 \phi(w - x) dw \right]$
= $xy^{x-1} (x^2 + 1),$

which is also a valid density on $0 < y < (x^2 + 1)^{-1/x}$.

Also, it follows from (9)

$$\frac{\partial^2}{\partial x \partial y} \log f(y|x) = \frac{1}{y} = E_{W|x} \left(\frac{\partial^2}{\partial x \partial y} \log f(y|x, W) \right).$$

Thus, average collapsibility holds, though the condition (i) is not satisfied.

It was quite challenging to construct Example 2, as it requires the interchange of log and integration, in addition to the other conditions. Observe also that in Example 2,

$$\frac{\partial}{\partial y} \log f(y|x, w) = \frac{\partial}{\partial y} \log f(y|x)$$
, for all (y, x) ,

which leads to the average collapsibility. This observation leads to the following result which generalizes Theorem 2 whose proof is immediate.

Theorem 3 The MDI is average collapsible over W if either

(i)
$$\frac{\partial}{\partial y} \log f(y|x, w) = \frac{\partial}{\partial y} \log f(y|x)$$
, for all (y, x) , or

(ii)
$$\frac{\partial}{\partial x} \log f(y|x, w) = \frac{\partial}{\partial x} \log f(y|x)$$
, for all (y, x)

holds.

As additional examples for Theorem 3, let f(y|x, w) be as in Example 2, consider, for $\lambda > 0$, the tempered normal density

$$t_{\lambda}(w|x) = c_{\lambda}(x)e^{-\lambda w}\phi(w-x)$$
, for $x > 0, w \in \mathbb{R}$,

where

$$c_{\lambda}(x) = \left(\int_{-\infty}^{\infty} e^{-\lambda w} \phi(w - x) dw \right)^{-1} = e^{(x^2 - (x - \lambda)^2)/2}.$$

That is, $t_{\lambda}(w|x) = \phi(w - x + \lambda)$. Then the corresponding marginal density of (Y|x) is

$$f_{\lambda}(y|x) = xy^{x-1} \left[\int_{-\infty}^{\infty} x^2 \phi(w - x + \lambda) dw + \int_{-\infty}^{\infty} (w - x)^2 \phi(w - x + \lambda) dw \right]$$

= $xy^{x-1} (x^2 + \lambda^2 + 1),$

which is also a valid density on $0 < y < (x^2 + \lambda^2 + 1)^{-1/x}$. Thus, the average collapsibility of *MDI* holds for the family $\{t_{\lambda}(w|x)\}, \lambda > 0$, also.

Next, we discuss the connection of Theorem 2 to logistic regression models.

Logistic regression. Let *Y* be binary and consider the following conditional and marginal logistic regression models (Vellaisamy and Vijay (2007), Xie *et al.* (2008)) considered in the literature:

$$\log\left(\frac{f(1|x,w)}{f(0|x,w)}\right) = \begin{cases} \alpha(w) + \beta(w)x, & \text{if } W \text{ is discrete} \\ \alpha + \beta x + \gamma w, & \text{if } W \text{ is continuous} \end{cases}$$

and

$$\log\left(\frac{f(1|x)}{f(0|x)}\right) = \tilde{\alpha} + \tilde{\beta}x.$$

We say the logistic regression coefficient is simply collapsible if

$$\tilde{\beta} = \begin{cases} \beta(w) \text{ for all } w, & \text{if } W \text{ is discrete} \\ \beta, & \text{if } W \text{ is continuous.} \end{cases}$$

Also, we say $\beta(w)$ or β is said to be average collapsible if $E_{W|x}(\beta(W)) = \tilde{\beta}$, when W is discrete and $E_{W|x}(\beta) = \tilde{\beta}$, when W is continuous.

Since Y is binary, the partial derivative is replaced by the difference between the adjacent levels of Y (see Cox (2003)) so that

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \log f(y|x, w) \right) = \frac{\partial}{\partial x} (\log f(1|x, w) - \log f(0|x, w))$$

$$= \frac{\partial}{\partial x} \log \left(\frac{f(1|x, w)}{f(0|x, w)} \right)$$

$$= \begin{cases} \frac{\partial}{\partial x} (\alpha(w) + \beta(w)x) = \beta(w), & \text{if } W \text{ is discrete} \\ \frac{\partial}{\partial x} (\alpha + \beta x + \gamma w) = \beta, & \text{if } W \text{ is continuous,} \end{cases}$$

the logistic regression coefficients corresponding to both the cases of W.

From Theorem 2, we now conclude that $\beta(w)$ or β is average collapsible if $(i)Y \perp \!\!\! \perp W|X$ or $(ii)X \perp \!\!\! \perp W|Y$ holds.

Finally, we discuss a new measure called log-expectation dependence (*LED*) between *X* and *Y* > 0, defined by $\partial \log E(Y|x, w)/\partial x$, where it is assumed that $0 < E(Y|x) < \infty$, for all *x*. First note that for all *x*,

$$\frac{\partial}{\partial x} \log E(Y|x) = 0 \iff \frac{\partial}{\partial x} E(Y|x) = 0$$

$$\iff \int y \frac{\partial}{\partial x} (dF(y|x)) = 0$$

$$\iff dF(y|x) = dF(y|x') \text{ for all } y, x \text{ and } x'$$

$$\iff Y \perp \!\!\!\perp X.$$

Also, by Theorem 1 of Xie et al. (2008),

$$\frac{\partial}{\partial x} \log E(Y|x) \ge 0 \Longrightarrow \frac{\partial}{\partial x} E(Y|x) \ge 0 \Longrightarrow \rho(Y,X) \ge 0,$$

where $\rho(Y, X)$ is the correlation coefficient between Y and X.

Next, we discuss the collapsibility issues for the *LED* measure and hence the following definition.

Definition 3 *The LED is simple collapsible if*

$$\frac{\partial}{\partial x} \log E(Y|x, w) = \frac{\partial}{\partial x} \log E(Y|x), \quad \text{for all } x \text{ and } w$$
 (10)

and average collapsible if

$$E_{W|x}\left(\frac{\partial}{\partial x}\log E(Y|x,W)\right) = \frac{\partial}{\partial x}\log E(Y|x), \quad \text{for all } x. \tag{11}$$

Theorem 4 The LED is simple collapsible and hence average collapsible if E(Y|x, w) does not depend on w.

We next discuss relevance of *LED* in the context of Poisson and negative binomial (NB) regression models. **Poisson regression**. Consider the Poisson regression model defined by

$$(Y|X = x, W = w) \sim Poi(\lambda(x, w)),$$

where the mean

$$E(Y|x, w) = \lambda(x, w) = \begin{cases} e^{\alpha(w) + \beta(w)x}, & \text{if } W \text{ is discrete} \\ e^{\alpha + \beta x + \gamma w}, & \text{if } W \text{ is continuous.} \end{cases}$$

Then

$$\frac{\partial}{\partial x} (\log E(Y|x, w)) = \begin{cases} \beta(w), & \text{if } W \text{ is discrete} \\ \beta, & \text{if } W \text{ is continuous.} \end{cases}$$

Let $(Y|x) \sim Poi(e^{\tilde{\alpha}+\tilde{\beta}x})$, the marginal Poisson regression model, so that

$$\log E(Y|x) = \tilde{\alpha} + \tilde{\beta}x; \quad \frac{\partial}{\partial x} \log E(Y|x) = \tilde{\beta}.$$

Then by Theorem 4, the average collapsibility of Poisson regression coefficient $\beta(w)$ (or β) holds, that is,

$$E_{W|x}(\beta(W)) = \tilde{\beta} \quad (or \ E_{W|x}(\beta) = \tilde{\beta})$$

is true, when $\lambda(x, w)$ does not depend on w which in turn holds when for example $\gamma = 0$. Note that this does not in general mean that $Y \perp \!\!\! \perp W | X$.

The following interesting example shows that average collapsibility may hold, even when E(Y|x,w) depends on w.

Example 3 Let X > 0 and $(Y|x, w) \sim P(\lambda(x)w)$, where $\lambda(x) = exp(\alpha + \beta x)$. Then $E(Y|x, w) = \lambda(x)w$ and

$$\frac{\partial}{\partial x} \log E(Y|x, w) = \beta = E_{W|x} \left(\frac{\partial}{\partial x} \log E(Y|x, W) \right). \tag{12}$$

Let now $(W|x) \sim G(x, x)$, the gamma distribution with mean unity. Then it is known that

$$(Y|x) \sim NB\left(x, \frac{x}{x + \lambda(x)}\right),$$

the negative binomial (NB) distribution with

$$P(Y = y|x) = \frac{\Gamma(y+x)}{y! \Gamma(x)} \left(\frac{x}{x+\lambda(x)}\right)^x \left(\frac{\lambda(x)}{x+\lambda(x)}\right)^y, \ y = 0, 1, \dots$$

Hence,

$$E(Y|x) = \lambda(x); \ \frac{\partial}{\partial x} \log E(Y|x) = \beta.$$
 (13)

Thus, from (12) and (13), the average collapsibility holds. Note here the covariates W and X are not independent.

Negative binomial regression. Suppose in Example 3 we assume in addition that the unobservable W is independent of X and $W \sim G(\theta, \theta)$. Then again

$$(Y|x) \sim NB\left(\theta, \frac{\theta}{\theta + \lambda(x)}\right); \quad E(Y|x) = \lambda(x).$$
 (14)

The model (14) is the usual NB regression model. Thus, the average collapsibility of the LED function corresponds to that of the NB regression coefficient β . It is interesting to note that when the unobserved covariate W follows the gamma distribution with mean unity, the average collapsibility of the NB regression coefficient holds, even when W and X are not independent (Example 3). Note, however, in the negative binomial regression,

$$Var(Y|x) = \lambda(x) \left(1 + \frac{\lambda(x)}{\theta} \right) > \lambda(x) = E(Y|x),$$
 (15)

unlike the Poisson regression case. Thus, whenever the data exhibits over dispersion (variance exceeds mean), the negative binomial regression model is commonly used.

3. The Multivariate Case

In this section, we consider an extension to the multivariate case. The case of multivariate response Y may be considered by treating one component at a time (Cox and Wermuth (2003) and Xie *et al.* (2008)) and similarly the covariate X may also be considered one component at a time, while keeping other components fixed. Therefore, we consider here only the case of multivariate random vector $W = (W_1, \ldots, W_p)$.

A conditional measure of association, say, $\frac{\partial}{\partial x}(E(Y|x,w))$ is simple collapsible over W if

$$\frac{\partial}{\partial x}(E(Y|x,w)) = \frac{\partial}{\partial x}(E(Y|x)), \text{ for all } x \text{ and } w = (w_1, \dots, w_p).$$

and average collapsible if

$$E_{W|x}\left(\frac{\partial}{\partial x}(E(Y|x,W))\right) = \frac{\partial}{\partial x}(E(Y|x)), \text{ for all } x.$$

The definition of average collapsibility of other measures of association remains the same, except that W is now a p- variate random vector.

Let $W = (W_1, W_2)$, where W_1 has q components and W_2 has (p-q) components. We now have the following result for the EDF and MDI and the corresponding results for LED follow easily when E(Y|x, w) is homogeneous over w.

Theorem 5 Let $W_1 \perp \!\!\! \perp W_2 | X$. Then the following results hold:

- (a) The EDF is average collapsible over W if (i) $Y \perp W_1 \mid (X, W_2)$ and (ii) $X \perp W_2$ hold.
- (b) The MDI is average collapsible over W if (i) $Y \perp W_1 \mid X$ and (ii) $X \perp W_2 \mid Y$ hold.

By symmetry, the average collapsibility of *MDI* holds when *X* and *Y* are interchanged in conditions (*i*) and (*ii*) of Part (b) of Theorem 5. Also, Xie *et al.* (2008) established the uniform collapsibility of *DDF* and *EDF* under an additional condition of homogeneity of these measures. Thus, average collapsibility holds under less restrictive conditions and hence is applicable to a larger class of conditional distributions that may arise in practical applications.

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APPENDIX

Proof of Theorem 1. Note that

$$\frac{\partial}{\partial x}E(Y|x) = \frac{\partial}{\partial x}\int E(Y|x,w)f(w|x)dw
= \int \frac{\partial}{\partial x}E(Y|x,w)f(w|x)dw + \int E(Y|x,w)\frac{\partial}{\partial x}f(w|x)dw
= E_{W|x}\left(\frac{\partial}{\partial x}E(Y|x,W)\right) + \int E(Y|x,w)\frac{\partial}{\partial x}f(w|x)dw.$$
(A.1)

Hence, average collapsibility holds if and only if

$$\int E(Y|x, w) \frac{\partial}{\partial x} f(w|x) dw = 0, \quad \text{for all } x.$$
 (A.2)

Assume now condition (i) holds so that

$$E(Y|x, w) = h(x)$$
, for all x and w, (say).

Then

$$\int E(Y|x, w) \frac{\partial}{\partial x} f(w|x) dw = h(x) \int \frac{\partial}{\partial x} f(w|x) dw$$
$$= h(x) \frac{\partial}{\partial x} \int f(w|x) dw$$
$$= 0, \text{ for all } x.$$

Hence, average collapsibility holds.

Assume next condition (ii) holds. Then obviously,

$$\int E(Y|x, w) \frac{\partial}{\partial x} f(w|x) dw = 0, \quad \text{for all } x,$$

and so average collapsibility holds again.

Proof of Theorem 2 Since

$$\frac{\partial^2}{\partial x \partial y} \log f(y|x) = \frac{\partial}{\partial x} \left(\frac{\frac{\partial}{\partial y} f(y|x)}{f(y|x)} \right), \tag{A.3}$$

average collapsibility of MDI holds if and only if

$$E_{W|x}\left(\frac{\partial}{\partial x}\left(\frac{\frac{\partial}{\partial y}f(y|x,W)}{f(y|x,W)}\right)\right) = \frac{\partial}{\partial x}\left(\frac{\frac{\partial}{\partial y}f(y|x)}{f(y|x)}\right) \text{ for all } (y,x). \tag{A.4}$$

Note that condition (i) implies

$$f(y|x, w) = f(y|x) \implies \frac{\partial}{\partial y} f(y|x, w) = \frac{\partial}{\partial y} f(y|x), \text{ for all } (y, x, w).$$

Thus, equation (A.4) holds.

Observe also that

$$\frac{\partial^2}{\partial x \partial y} \log f(x, y) = \frac{\partial}{\partial y} \left(\frac{\frac{\partial}{\partial x} (x|y)}{f(x|y)} \right),$$

which is the same as equation (A.3) with x and y interchanged. Thus, the condition (ii) also implies the average collapsibility of MDI.

Proof of Theorem 4 Let

$$E(Y|x, w) = h_1(x) \quad \text{for all } x \text{ and } w. \tag{A.5}$$

Then

$$E(Y|x) = E_{W|x}(E(Y|x, W)) = E_{W|x}(h_1(x)) = h_1(x).$$
(A.6)

Thus, from (A.5) and (A.6),

$$E(Y|x) = E(Y|x, w)$$
, for all x and w,

and hence simple collapsibility holds.

Also, since

$$\frac{\partial}{\partial x} \log E(Y|x, w) = \frac{\partial}{\partial x} \log E(Y|x)$$

the average collapsibility also holds.

Proof of Theorem 5 (a) Observe that

$$E_{W|x}\left(\frac{\partial}{\partial x}E(Y|x,W)\right) = \int_{w_2} \int_{w_1} \left(\frac{\partial}{\partial x}E(Y|x,w)\right) dF(w_1,w_2|x)$$

$$= \int_{w_2} \int_{w_1} \left(\frac{\partial}{\partial x}E(Y|x,w_1,w_2)\right) dF(w_1|x) dF(w_2|x), \quad (\because W_1 \perp \!\!\!\perp W_2|X)$$

$$= \int_{w_2} \left(\int_{w_1} \frac{\partial}{\partial x}E(Y|x,w_2) dF(w_1|x)\right) dF(w_2|x), \quad (\because Y \perp \!\!\!\perp W_1|(X,W_2))$$

$$= \int_{w_2} \left(\frac{\partial}{\partial x}E(Y|x,w_2)\right) dF(w_2|x)$$

$$= E_{W_2|x} \left(\frac{\partial}{\partial x}E(Y|x,W_2)\right)$$

$$= \frac{\partial}{\partial x}E(Y|x) \quad \text{for all } x,$$

by condition (ii) of (a) and Theorem 1.

(b) First observe that

$$\frac{\partial^2}{\partial x \partial y} \log f(x, y | w) = \frac{\partial^2}{\partial x \partial y} \log f(x, y, w_1, w_2)$$

$$= \frac{\partial^2}{\partial x \partial y} \log f(y | x, w_1, w_2)$$

$$= \frac{\partial^2}{\partial x \partial y} \log f(w | x, w_2), \tag{A.7}$$

since $Y \perp \!\!\! \perp W_1 | X$. By the assumption that $W_1 \perp \!\!\! \perp W_2 | X$ and (A.7),

$$E_{W|x}\left(\frac{\partial^{2}}{\partial x \partial y} \log f(x, y|W)\right) = \int_{w_{2}} \left(\int_{w_{1}} \frac{\partial^{2}}{\partial x \partial y} \log f(y|x, w_{2}) dF(w_{1}|x)\right) dF(w_{2}|x)$$

$$= \int_{w_{2}} \left(\frac{\partial^{2}}{\partial x \partial y} \log f(y|x, w_{2})\right) dF(w_{2}|x)$$

$$= E_{W_{2}|x} \left(\frac{\partial^{2}}{\partial x \partial y} \log f(y|x, W_{2})\right)$$

$$= \frac{\partial^{2}}{\partial x \partial y} \log f(y|x) \quad \text{for all } x \text{ and } y$$

by condition (ii) of Theorem 2.

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